

Notes on orbit and spin tracking in an electrostatic storage ring

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Abstract

Two documents have recently been posted on the arXiv describing a numerical integration algorithm: “symplectic orbit/spin tracking code for all-electric storage rings” [1] and some computational results therefrom [2]. This note comments critically on some of the claims in [1] and [2]. In particular, it is not clear that the orbit tracking algorithm described in [1] is really symplectic. Specifically, for electrostatic beamline elements, the so-called “zero length elements,” which are treated as position dependent kicks in the formalism in [1], are in fact *not* symplectic.

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Two documents have recently been posted on the arXiv [1, 2], the former describing a numerical integration algorithm with the claim “symplectic orbit/spin tracking code for all-electric storage rings” while the latter presents some computational results therefrom. The preprint [1] is essentially a description of a numerical integration algorithm for the orbit and spin motion in an all-electric storage ring, implemented in a new program “ETEAPOT” written by the authors. The work is motivated by efforts to build an electrostatic storage ring to circulate spin-polarized beams of charged hadrons such as protons (or other light nuclei) to search for a possible nonzero permanent electric dipole moment (EDM). It is standard practice to submit preliminary reports of work at conferences, e.g. [3] (presented at IPAC12), and to follow up with a full-length article(s) in a peer-review journal.

This note comments critically on some of the claims in [1] and [2]. In particular, it is not clear that the orbit tracking algorithm described in [1] is really symplectic. Specifically, for electrostatic beamline elements, it should be noted that “zero length elements,” which are treated as position dependent kicks in the numerical integration algorithm in [1], are in fact *not* symplectic. Note also a technical detail about Lie groups: only the orbital motion lies on a symplectic manifold; for spin the corresponding concept is “unitarity.” The spin is treated as a classical unit vector in [1] and [2], hence the pertinent Lie group is $SO(3)$.

The numerical integration algorithm in [1] seems to be exactly the same as that which led to the “cylindrical miracle” [4]. This was an unpublished report by the same authors, dated June 2014, where the tracking output over 10^5 turns demonstrated damping of the orbital oscillations, in violation of Liouville’s theorem. (The document [2] presents results where particles were tracked for 33 million turns and it is claimed that no damping of the orbits was observed. However the results presented in [2] did not analyze the same model as that treated in [4].) I tracked the orbital motion for the same model treated in [4]. I tracked for 10^6 turns, with the same inputs employed by the authors, and observed no damping of the orbital oscillations. As required for a Hamiltonian system, I found that the phase space volume was conserved and the Courant-Snyder invariant of the radial oscillations did not damp. For this reason, it is not clear that the numerical integration algorithm (for the orbit) in [1] is really symplectic.

The bibliography in [1] is curious: there are citations to papers and/or books and/or unpublished reports where one or both of J. and R. Talman are coauthor(s), but there are

no references to other papers published by the SREDM (Storage Ring EDM) collaboration, where J. and R. Talman are not coauthors. For example there is no citation to a recent paper on “precision results” for benchmark tests using a different program, a fourth order Runge-Kutta integrator [5]. (I have published a Comment paper [6] pointing out loopholes in the analysis in [5]. The document [2] does not present tracking results for the benchmark tests listed in [5] or additional tests listed in [7].) However, [1] does cite a manuscript submitted to *Physical Review Letters* by the SREDM collaboration (Ref. 1 in [1]: ““Proton EDM group, Storage Ring Electric Dipole Measurement,” paper in preparation for submission to PRL.” Both J. and R. Talman are coauthors.) That manuscript has been rejected by *Physical Review Letters*.

This note comments mainly on the numerical integration algorithm in [1]. (The preprint [2] is mainly concerned with tracking for a reconstruction of the Brookhaven Electron Analogue ring [8].) For clarity of the exposition, I summarize the numerical integration algorithm in [1] first and list additional comments afterwards. As is standard in the field, only isoelectric rings are treated. The reference bend radius is r_0 . The reference orbit lies in the horizontal (median) plane and the independent variable is the arc-length s along the reference orbit. The orbital coordinates are (x, y, z) where x is radial, y is vertical and z is along the reference orbit (we can say “longitudinal”). A positive bend is to the right, i.e. clockwise. The particle mass and charge are m_p and e , respectively. The momentum is $\mathbf{p} = \gamma m_p \mathbf{v}$, where \mathbf{v} is the velocity and γ is the Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$. The spin \mathbf{s} is treated as a classical unit vector.

The authors in [1] note that the exact solution is known in closed form for the orbit (and spin precession) in the relativistic Kepler problem, i.e. a Coulomb potential $V \propto 1/r$, where r is the radial distance from the center of curvature $r = \sqrt{(r_0 + x)^2 + y^2 + z^2}$. The solution for the orbit in the relativistic Kepler problem has been derived by several authors, e.g. [9–13]. The authors in [1] employ the solution in [13]. The solution for the spin precession was derived by the authors themselves in earlier publications. See eqs. (133)–(136) in [1]. It is well known that for any central force potential $V(r)$, the orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is conserved and the orbital motion lies in a plane normal to \mathbf{L} . The spin also precesses around an axis parallel to \mathbf{L} . The authors in [1] therefore begin by integrating the orbit and spin through a so-called “spherical bend,” i.e. a beamline element with a central

force potential, specifically a Coulomb potential. For example, for the spin, the entrance spin components $(s_x, s_y, s_z)_{\text{in}}$ are transformed (rotated) into components (s_1, s_2, s_3) , referenced to basis vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, where $\mathbf{e}_3 \parallel \mathbf{p}$ and $\mathbf{e}_2 \parallel -\mathbf{L}$ and $\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3$. (See eq. (126) in [1].) Then the component s_2 is invariant through the spherical bend while s_1 and s_3 precess around \mathbf{e}_2 . At the exit of the element, the spin components are rotated back to the (x, y, z) coordinate system, to obtain exit values $(s_x, s_y, s_z)_{\text{out}}$. The solution for the orbit is derived in Sections II – VI in [1]. That solution also involves transformations between the entrance/exit (x, y, z) coordinate system and the variables in the bend plane normal to \mathbf{L} . Note that the orbital angular momentum \mathbf{L} is different for each particle, so the above coordinate transformations must be performed individually for each particle.

In general, of course, the potential in an electrostatic bend is not a Coulomb potential. First define $\rho = r_0 + x$. (I avoid the use of r to avoid confusion with the three-dimensional usage for the relativistic Kepler problem.) The electric field \mathbf{E} in the median plane is given in terms of a field index m (see eq. (2) in [1])

$$\mathbf{E} = -E_0 \frac{r_0^{1+m}}{\rho^{1+m}} \hat{\mathbf{x}}. \quad (1)$$

Here E_0 is a constant. The Coulomb potential is given by $m = 1$, but in general $0 \leq m < 1$. The case $m = 0$ corresponds to a cylindrical capacitor, where the electric field is purely radial and there is no vertical focusing. Section VII of [1] states their procedure succinctly: “The ETEAPOT strategy, even for $m \neq 1$, is to treat sector bends as thick elements with orbits given by the analytic $m = 1$ formulas. The $m \neq 1$ case is handled by inserting zero thickness “artificial quadrupoles” of appropriate strength.” (The zero length quadrupoles would also act on the spin.) Essentially, a sector bend is sliced and zero length quadrupoles are inserted between the slices. By definition, the electric field on the reference orbit is chosen so that the reference orbit is a circle of radius r_0 . It is the gradient, and higher derivatives, of the electric field which depend on the field index. For a sector bend with a field index m , Hill’s equations for the focusing in the horizontal and vertical directions are [14]

$$\frac{d^2x}{ds^2} = -\frac{2 - m - \beta_0^2}{r_0^2} x, \quad \frac{d^2y}{ds^2} = -\frac{m}{r_0^2} y. \quad (2)$$

As stated above, a Coulomb potential corresponds to $m = 1$. Hence the zero length quadrupoles are chosen to modify the focusing to yield eq. (2). Writing $d^2x/ds^2 = -\Delta K_x x$

and $d^2y/ds^2 = -\Delta K_y y$ where $\Delta K_{x,y}$ are the horizontal and vertical focusing gradients of the artificial quadrupoles, the authors state (eq. (121) in [1])

$$\Delta K_x = \frac{1-m}{r_0^2}, \quad \Delta K_y = \frac{m-1}{r_0^2}. \quad (3)$$

As is standard for lumped (zero length) elements, the integrated quadrupole gradient of an artificial quadrupole is nonzero. From Section VII in [1]: “The artificial quadrupoles have zero length, but their length-strength product has to be matched to the “field integral” corresponding to length L_{bend} of the bending slice being compensated.”

In addition, the authors in [1] also describe integration of the orbit and spin through so-called “thin” elements, i.e. lumped elements of zero physical length but nonzero integrated focusing (or higher) gradient(s). This includes quadrupoles and all higher multipoles. From Section VIII in [1]: “In ETEAPOT the only thick elements are bends and drifts. ... All other elements are treated as thin element (position dependent) kicks.” The term “position dependent kicks” implies an impulse change to the momentum (and no change to the orbital coordinates) and a finite (or non-infinitesimal) spin rotation (see eq. (147) in [1]).

- It is stated in [1] that the use of thin elements is symplectic. This is not true and is a serious issue. From Section VII in [1] (italics mine): “The ETEAPOT strategy, even for $m \neq 1$, is to treat sector bends as thick elements with orbits given by the analytic $m = 1$ formulas. The $m \neq 1$ case is handled by inserting zero thickness “artificial quadrupoles” of appropriate strength. ... Though the idealized model differs from the physical apparatus, *the orbit description within the idealized model is exact, and hence symplectic.*” For electrostatic fields, the treatment of thin elements as position dependent kicks is not symplectic. We can elucidate the above point as follows. The canonical variables for motion in a storage ring, using the arc-length s as the independent variable, are $(x, p_x, y, p_y, -t, H)$, where H is the total energy and t is the time of arrival of a particle. Note that γ is *not* a dynamical variable in the presence of electrostatic fields. For an all-electric ring, the Hamiltonian is (neglecting rf cavities)

$$K_{\text{elec}} = -\left(1 + \frac{x}{r_0}\right) \left[\frac{(H - eV)^2}{c^2} - m_p^2 c^2 - p_x^2 - p_y^2 \right]^{1/2}. \quad (4)$$

The electrostatic potential V depends only on the coordinates (x, y) and s . The potential V appears *inside* the square root. It is not in general valid to treat the potential V

as a separate term from the rest of the Hamiltonian, i.e. as a purely position dependent kick, even for so-called “thin elements.”

- The above issues stem from a significant weakness in [1], viz. the lack of the use of a proper set of canonical dynamical variables (coordinates and conjugate momenta), as is required to truly obtain a symplectic description of the orbital motion. (I have attempted to resolve these issues privately with R. Talman, but he has not replied.) A Hamiltonian in fact never appears in [1]; specifically, no Hamiltonian is specified for the so-called “position dependent kicks.”
- Following from the above points, it is stated in Section I in [1] (boldface in original): **“Complications imposed by electric bending.** The fundamental complication of an electric ring, as contrasted to a magnetic ring, is the non-constancy of particle speed. A fast/slow separation into betatron and synchrotron amplitudes has become fundamental to the conventional (Courant-Snyder) magnetic ring formalism.” In fact, the Courant-Snyder formalism is valid for both electrostatic and magnetic storage rings. The third momentum in both cases is the total energy H , or an offset $\Delta H/H_0$, i.e. the orbital motion should be expressed using canonical dynamical variables. In a magnetic ring $\Delta H/H_0 = \Delta\gamma/\gamma_0$, but this not the case in an electric ring, but this fact has no relevance to the applicability of the Courant-Snyder formalism. Also, for coasting beams (no rf), the total energy H (or the offset $\Delta H/H_0$) is a dynamical invariant, in both magnetic and electric rings.
- The following might be a misprint in [1]: the authors state in Section VII in [1] (italics mine): “Before continuing with the treatment for $m \neq 0$ it is important to remember *the discontinuous increments to \mathcal{E} as a particle enters or leaves a bending element.* The discontinuity is equal (in magnitude) to the change in potential energy.” However, \mathcal{E} is defined as the total energy (kinetic plus potential) in Section I in [1]. By conservation of energy, therefore, the value of \mathcal{E} does *not* change for particle motion in an electrostatic field, including the traversal of fringe fields at the entrance and exit of a beamline element. Perhaps the authors mean to say that the value of γ changes discontinuously as a particle enters or leaves a bending element, which is indeed the case for a hard edge model of the fringe fields.

- The preprint [2] contains various documents of historical interest concerning the BNL Electron Analogue ring (a 10 MeV electrostatic electron synchrotron [8]) and presents some tracking outputs using a model reconstruction of that ring.
 - Since the tracking algorithm described in [1] appears to be the same as that in previous work by the same authors, the “cylindrical miracle” [4] may still exist in their present formalism. The tracking simulations in [2] do not treat the model previously studied by the authors in [4].
 - The preprint [2] also does not present results where the tracking outputs can be compared with known analytical formulas, e.g. the benchmark tests listed by other members of the SREDM collaboration [5] or those by myself in [6, 7].
 - The preprint [2] also does not contain tracking results to validate the formula for the spin coherence time derived by the authors themselves in [3]. This would be a significant self-consistency check of their formalism.

All of the above tests can be interpreted as suggestions for future work and would be significant checks of the authors’ formalism in [1].

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